Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 1, pp. 107–116, 1998

Printed in Great Britain
0017–9310/98 $19.00+0.00

PII: S0017–9310(97)00098–7

Hyperbolic thermal waves in a solid cylinder with a non-stationary boundary heat flux

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(*Received* 20 *January* 1997 *and in final form* 13 *March* 1997)

Abstract—The non-stationary heat conduction in an infinitely long solid cylinder with a time-dependent boundary heat flux is studied for a material with a non-vanishing thermal relaxation time. An analytical solution of the hyperbolic energy equation together with its boundary and initial conditions is obtained by the Laplace transform method. The temperature distribution and the heat flux density distribution are studied both for a constant boundary heat flux and for an exponentially decaying boundary heat flux. The compatibility of these distributions with the local equilibrium hypothesis is analysed. © 1997 Elsevier Science Ltd.

INTRODUCTION

In the last decades, one of the most interesting results obtained in the field of heat transfer in solid media is the evidence that discrepancies between Fourier's law and experimental results may occur in highly non-stationary phenomena. Experimental analyses of the inadequacy of Fourier's law, when very rapid changes of temperature or of heat flux occur, were performed both at very low temperatures [1, 2] and at room temperatures [3, 4]. When Fourier's law cannot be applied, a theoretical description of the heat conduction process can be provided by the Cattaneo–Vernotte constitutive equation [5–7]

$$\mathbf{q}+\tau\frac{\partial \mathbf{q}}{\partial t} = -k\nabla T. \quad (1)$$

Equation (1) reduces to Fourier's law when the term $\tau\,\partial\mathbf{q}/\partial t$ can be neglected with respect to $\mathbf{q}$. In other words, equation (1) predicts that Fourier's law holds when relevant changes of the heat flux density occur only in time intervals much greater than τ. The thermal relaxation time τ is a property of the material and is estimated to range from 10^{-14} s for aluminium at high temperatures to 10^{3} s for biological tissue at cryogenic temperatures [8].

For a solid with constant mass density and such that $\mathrm{d}u = c\,\mathrm{d}T$, the local energy balance equation can be written as

$$\nabla\cdot\mathbf{q}+\rho c\frac{\partial T}{\partial t} = 0. \quad (2)$$

As a consequence of equations (1) and (2), the temperature field of a solid with constant values of ρ, c, k and τ must fulfil the differential equation

$$\alpha\nabla^2 T = \frac{\partial T}{\partial t}+\tau\frac{\partial^2 T}{\partial t^2}. \quad (3)$$

Equation (3) is a hyperbolic equation which predicts that a temperature change propagates as a damped wave with a speed $\sqrt{\alpha/\tau}$.

Many solutions of equation (3) for plane thermal waves are available in the literature. On the other hand, few papers deal with the propagation of cylindrical thermal waves. In particular, Wilhelm and Choi [9] describe the evolution of the temperature field in an infinite solid medium starting from a delta-like temperature distribution centred on a line. Hector *et al.* [10] study the hyperbolic heat conduction in an infinitely wide plane slab with an insulated surface and a heated surface. They consider a non-stationary and axisymmetric distribution of the heat flux density on the heated surface. In ref. [11], the onset of resonance phenomena for the temperature field in a cylindrical electric resistor carrying a steady-periodic current is analysed. In ref. [12], the thermal waves generated in an infinite solid medium by a cylindrical hot wire which provides a non-stationary heat flux are studied. For this system, it is shown that the temperature field can violate the hypothesis of local thermodynamic equilibrium [12].

The aim of the present paper is to study the temperature field and the heat flux density distribution in an infinitely long solid cylinder with a non-stationary boundary heat flux. Particular attention will be devoted to the case of a step change of the boundary heat flux and to the case of an exponentially decaying boundary heat flux. The following novel feature of hyperbolic heat conduction will be pointed out. When a sudden and finite change of the heat flux density is prescribed at the boundary of a solid cylinder, both the temperature field and the heat flux density present

NOMENCLATURE

- a_n dimensionless coefficients employed in the Appendix
- $B(\xi, \omega, \Lambda)$ dimensionless function defined by equation (27)
- c specific heat [J (kg K)$^{-1}$]
- $g(\eta)$ dimensionless function employed in the Appendix
- $F(t)$ $= q(r_0, t)/q_0$, dimensionless function
- I_v modified Bessel function of first kind and order v
- J_v Bessel function of first kind and order v
- i $= \sqrt{-1}$, imaginary unit
- k thermal conductivity [W (m K)$^{-1}$]
- n non-negative integer number
- p Laplace transformed variable
- $\mathbf{q}$ heat flux density [W m^{-2}]
- q radial component of $\mathbf{q}$ [W m^{-2}]
- q_0 reference value of the heat flux density [W m^{-2}]
- Q_∞ heat supplied to the cylinder per unit length in the time interval $[0, +\infty]$, [J m^{-1}]
- Res{ ; } residue of a complex analytic function at a pole
- Re{ } real part of a complex number
- r radial coordinate [m]
- r_0 radius of the cylinder [m]
- s entropy per unit mass [J (kg K)$^{-1}$]
- t time [s]
- t_p constant time employed in equation (49) [s]
- T temperature [K]
- T_0 initial temperature [K]
- u internal energy per unit mass [J kg^{-1}]
- U unit step function
- y real variable employed in the Appendix.

Greek letters

- α thermal diffusivity [m^2 s^{-1}]
- β_n nth real non-negative root of equation $J_1(\beta) = 0$
- β $= i\sqrt{p + \Lambda p^2}/2$, complex variable
- γ constant employed in the inversion formula (20)
- ζ complex variable employed in equation (24)
- η $= r/r_0$, dimensionless radial coordinate
- ϑ dimensionless temperature defined in equation (8)
- Λ $= \alpha\tau/(4r_0^2)$, dimensionless thermal relaxation time
- λ_n, μ_n dimensionless parameters defined in equation (22)
- ξ $= \alpha t/(4r_0^2)$, dimensionless time
- Π dimensionless parameter defined in equation (50)
- ρ mass density [kg m^{-3}]
- σ entropy production rate per unit volume [W (m^{-3} K^{-1})]
- τ thermal relaxation time [s]
- Φ dimensionless function of ξ and Λ defined by equation (11)
- χ dimensionless heat flux density defined in equation (8)
- ω complex variable employed in equation (27).

Superscripts

- $\tilde{}$ Laplace transformed function
- $'$ dummy integration variable.

singularities. Moreover, an analysis of the compatibility of the solutions of equation (3) with the local equilibrium hypothesis will be performed by means of the method pointed out in ref. [12] and widely discussed in ref. [13]. More severe checks on the validity of local equilibrium could be performed according to the physical interpretation of hyperbolic heat conduction proposed in the theory of extended irreversible thermodynamics (EIT) [14, 15]. This theory is based on a generalized thermodynamic scheme which does not employ the hypothesis of local equilibrium and predicts a differential equation for the temperature field more general than equation (3). An analysis of the compatibility of the solutions of equation (3) with the local equilibrium hypothesis according to EIT is beyond the purposes of this paper.

GOVERNING EQUATIONS

In this section, the hyperbolic energy equation and its initial and boundary conditions are written in a dimensionless form. Then, the dimensionless temperature field within the cylinder is obtained by employing the Laplace transform method. Let us consider an infinitely long solid cylinder with radius r_0. Let us assume that the density, the thermal conductivity, the thermal diffusivity and the thermal relaxation time of the cylinder are constant. At time $t = 0$, the temperature within the cylinder is uniform with value T_0, while the heat flux density distribution is zero. As a consequence, at time $t = 0$ also $\partial T/\partial t$ is zero, as it is easily proved by employing equation (2). For $t > 0$, a uniform and time-dependent heat flux

$q(r_0, t) = -q_0 F(t)$ crosses radially the surface $r = r_0$, where $F(t)$ is an arbitrary dimensionless function of time such that $F(0) = 0$. Therefore, the temperature field within the cylinder must be axisymmetric, so that equation (3) can be rewritten as

$$\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2}. \quad (4)$$

The initial and boundary conditions can be expressed as

$$T(r, 0) = T_0 \quad \left.\frac{\partial T}{\partial t}\right|_{t=0} = 0 \quad (5)$$

$$k\left.\frac{\partial T}{\partial t}\right|_{r=r_0} = q_0\left[F(t) + \tau\frac{dF(t)}{dt}\right] \quad t > 0. \quad (6)$$

Since the temperature field must be regular on the axis of the cylinder, an additional constraint is given by

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0. \quad (7)$$

By employing the dimensionless quantities

$$\vartheta = k\frac{T - T_0}{r_0 q_0} \quad \chi = \frac{q}{q_0} \quad \eta = \frac{r}{r_0} \quad \xi = \frac{\alpha t}{4r_0^2} \quad \Lambda = \frac{\alpha\tau}{4r_0^2} \quad (8)$$

equations (4)–(7) yield

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \vartheta}{\partial \eta}\right) = \frac{1}{4}\left(\frac{\partial \vartheta}{\partial \xi} + \Lambda\frac{\partial^2 \vartheta}{\partial \xi^2}\right) \quad (9)$$

$$\vartheta(\eta, 0) = 0 \quad \left.\frac{\partial \vartheta}{\partial \xi}\right|_{\xi=0} = 0 \quad (10)$$

$$\left.\frac{\partial \vartheta}{\partial \eta}\right|_{\eta=1} = F(4r_0^2\xi/\alpha) + \Lambda\frac{dF(4r_0^2\xi/\alpha)}{d\xi} \equiv \Phi(\xi, \Lambda) \quad (11)$$

$$\left.\frac{\partial \vartheta}{\partial \eta}\right|_{\eta=0} = 0. \quad (12)$$

The Laplace transform of $\vartheta(\eta, \xi)$ is given by

$$\tilde{\vartheta}(\eta, p) = \int_0^{+\infty} e^{-p\xi}\vartheta(\eta, \xi)\,d\xi. \quad (13)$$

On account of equation (10) and of the properties of Laplace transforms [16], equations (9), (11) and (12) can be rewritten as

$$\frac{d^2\tilde{\vartheta}}{d\eta^2} + \frac{1}{\eta}\frac{d\tilde{\vartheta}}{d\eta} - \frac{1}{4}(p + \Lambda p^2)\tilde{\vartheta} = 0 \quad (14)$$

$$\left.\frac{d\tilde{\vartheta}}{d\eta}\right|_{\eta=1} = \tilde{\Phi}(p, \Lambda) \quad (15)$$

$$\left.\frac{d\tilde{\vartheta}}{d\eta}\right|_{\eta=0} = 0. \quad (16)$$

The solution of equations (14)–(16) can be expressed as

$$\tilde{\vartheta}(\eta, p) = \tilde{\vartheta}_0(\eta, p)\tilde{\Phi}(p, \Lambda) \quad (17)$$

where $\tilde{\vartheta}_0(\eta, p)$ is given by

$$\tilde{\vartheta}_0(\eta, p) = \frac{2I_0(\eta\sqrt{p + \Lambda p^2}/2)}{I_1(\sqrt{p + \Lambda p^2}/2)\sqrt{p + \Lambda p^2}}. \quad (18)$$

As a consequence of the convolution theorem for inverse Laplace transforms [16], the dimensionless temperature field can be expressed as

$$\vartheta(\eta, \xi) = \int_0^{\xi} \Phi(\xi', \Lambda)\vartheta_0(\eta, \xi - \xi')\,d\xi'. \quad (19)$$

The inversion theorem for Laplace transforms [16] yields

$$\vartheta_0(\eta, \xi) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{p\xi}\tilde{\vartheta}_0(\eta, p)\,dp \quad (20)$$

where γ is any real number such that the function $\tilde{\vartheta}_0(\eta, p)$ is analytic in the half-plane $\mathrm{Re}\{p\} > \gamma$. On account of equation (18) and of the properties of Bessel functions [17], $\tilde{\vartheta}_0(\eta, p)$ can be rewritten as

$$\tilde{\vartheta}_0(\eta, p) = -\frac{J_0(\eta\beta)}{\beta J_1(\beta)} \quad (21)$$

where $\beta = i\sqrt{p + \Lambda p^2}/2$. Therefore, $\tilde{\vartheta}_0(\eta, p)$ has infinite simple poles for $p = \lambda_n$ and for $p = \mu_n$, where

$$\lambda_n = -\frac{1}{2\Lambda}(1 + \sqrt{1 - 16\Lambda\beta_n^2})$$

$$\mu_n = -\frac{1}{2\Lambda}(1 - \sqrt{1 - 16\Lambda\beta_n^2}) \quad n \geqslant 0 \quad (22)$$

and β_n is the nth real non-negative root of equation $J_1(\beta) = 0$. On account of the properties of Bessel functions [17], if $\xi - (1 - \eta)\sqrt{\Lambda}/2 > 0$, the integral on the right-hand side of equation (20) coincides with the integral of $e^{p\xi}\tilde{\vartheta}_0(\eta, p)$ on a semicircular closed path which lies to the left of the line $\mathrm{Re}\{p\} = \gamma$ and is centred at $p = \gamma$, provided that the radius of the semicircle tends to infinity. On the other hand, if $\xi - (1 - \eta)\sqrt{\Lambda}/2 < 0$, the integral on the right-hand side of equation (20) coincides with the integral of $e^{p\xi}\tilde{\vartheta}_0(\eta, p)$ on a semicircular closed path which lies to the right of the line $\mathrm{Re}\{p\} = \gamma$ and is centred at $p = \gamma$, provided that the radius of the semicircle tends to infinity. Therefore, on account of Cauchy's residue theorem, equation (20) can be rewritten as

$$\vartheta_0(\eta, \xi) = U(\xi - (1 - \eta)\sqrt{\Lambda}/2) \times \sum_{n=0}^{\infty} [\mathrm{Res}\{e^{p\xi}\tilde{\vartheta}_0(\eta, p); p = \lambda_n\} + \mathrm{Res}\{e^{p\xi}\tilde{\vartheta}_0(\eta, p); p = \mu_n\}]. \quad (23)$$

If $\xi-(1-\eta)\sqrt{\Lambda}/2 < 0$, equation (23) ensures that function $\vartheta_0(\eta,\xi)$ is zero. Indeed, on account of equation (8), the inequality $\xi-(1-\eta)\sqrt{\Lambda}/2 < 0$ holds if and only if $t\sqrt{\alpha/\tau} < r_0-r$. Since $\sqrt{\alpha/\tau}$ represents the speed of thermal waves, the condition $t\sqrt{\alpha/\tau} < r_0-r$ is verified whenever t is not sufficiently large for the thermal wave to travel from the boundary to the given position r. By employing equation (21) and the properties of Bessel functions [17], one can express the residue of $e^{p\xi}\tilde{\vartheta}_0(\eta,p)$ at any simple pole $p=\zeta$ as follows:

$$\mathrm{Res}\{e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,;p=\zeta\} = \lim_{p\to\zeta}\frac{8e^{p\xi}J_0(\eta\beta)}{(2\Lambda p+1)J_0(\beta)}. \tag{24}$$

It can be verified that, if $\xi-(1-\eta)\sqrt{\Lambda}/2 < 0$, the infinite sum which appears on the right-hand side of equation (23) is zero. Therefore, the term $U(\xi-(1-\eta)\sqrt{\Lambda}/2)$ in equation (23) can be omitted, so that equations (22)–(24) yield

$$\vartheta_0(\eta,\xi) = 8(1-e^{-\xi/\Lambda}) - 8\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\sqrt{1-16\Lambda\beta_n^2}}(e^{\lambda_n\xi}-e^{\mu_n\xi}). \tag{25}$$

The dimensionless temperature field $\vartheta(\eta,\xi)$ can be determined by means of equations (19) and (25) and can be expressed as

$$\vartheta(\eta,\xi) = 8[B(\xi,0,\Lambda)-B(\xi,1/\Lambda,\Lambda)] - 8\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\sqrt{1-16\Lambda\beta_n^2}} \times[B(\xi,-\lambda_n,\Lambda)-B(\xi,-\mu_n,\Lambda)] \tag{26}$$

where function $B(\xi,\omega,\Lambda)$ is defined as

$$B(\xi,\omega,\Lambda) = e^{-\omega\xi}\int_0^{\xi} e^{\omega\xi'}\Phi(\xi',\Lambda)\,d\xi'. \tag{27}$$

On account of equations (11) and (27), equation (26) can be rewritten as

$$\vartheta(\eta,\xi) = 8\int_0^{\xi} F(4r_0^2\xi'/\alpha)\,d\xi' - 8\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\sqrt{1-16\Lambda\beta_n^2}} \times[B(\xi,-\lambda_n,\Lambda)-B(\xi,-\mu_n,\Lambda)]. \tag{28}$$

Equation (22) implies that, for small values of Λ, one can employ the approximate expressions $\lambda_n \approx -1/\Lambda$ and $\mu_n \approx -4\beta_n^2$. A representation of the behaviour of λ_n/β_n^2 and of μ_n/β_n^2 for small values of Λ is reported in Fig. 1. This figure shows that, when $\Lambda\to 0$, λ_n/β_n^2 tends to $-\infty$, while μ_n/β_n^2 tends to -4. As a consequence, if Fourier's law holds, i.e. in the limit $\Lambda\to 0$, equation (28) yields

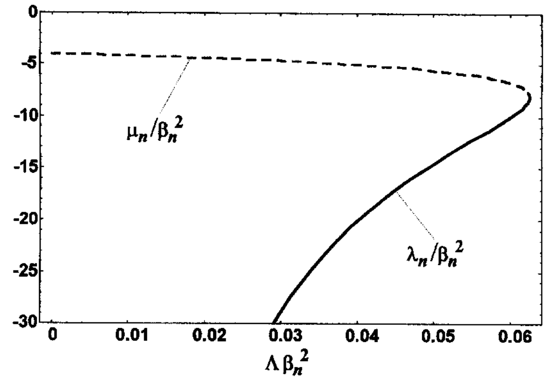

Fig. 1. Plots of λ_n/β_n^2 and μ_n/β_n^2 vs $\Lambda\beta_n^2$.

$$\vartheta(\eta,\xi) = 8\int_0^{\xi} F(4r_0^2\xi'/\alpha)\,d\xi' + 8\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)}B(\xi,4\beta_n^2,0). \tag{29}$$

THE HEAT FLUX DENSITY AND THE LOCAL EQUILIBRIUM HYPOTHESIS

In this section, an expression of the dimensionless heat flux density is determined. Then, a criterion to check the validity of the local equilibrium hypothesis is presented. As a consequence of equation (2), the local energy balance equation can be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}(rq)+\rho c\frac{\partial T}{\partial t} = 0. \tag{30}$$

On account of equation (8), equation (30) yields

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\chi) = -\frac{1}{4}\frac{\partial\vartheta}{\partial\xi} \tag{31}$$

so that the dimensionless heat flux density χ is given by

$$\chi(\eta,\xi) = -\frac{1}{4\eta}\int_0^{\eta}\frac{\partial\vartheta(\eta',\xi)}{\partial\xi}\eta'\,d\eta'. \tag{32}$$

Equation (27) implies the following identity:

$$\frac{\partial B(\xi,\omega,\Lambda)}{\partial\xi} = -\omega B(\xi,\omega,\Lambda)+\Phi(\xi,\Lambda). \tag{33}$$

By employing equations (28), (32) and (33), one obtains

$$\chi(\eta,\xi) = -\eta F(4r_0^2\xi/\alpha) + 2\sum_{n=1}^{\infty}\frac{J_1(\eta\beta_n)}{\beta_n J_0(\beta_n)\sqrt{1-16\Lambda\beta_n^2}} \times[\lambda_n B(\xi,-\lambda_n,\Lambda)-\mu_n B(\xi,-\mu_n,\Lambda)]. \tag{34}$$

In the limit $\Lambda \to 0$, the approximate expressions $\lambda_n \approx -1/\Lambda$ and $\mu_n \approx -4\beta_n^2$ hold, so that equation (34) yields

$$\chi(\eta,\xi) = -F(4r_0^2\xi/\alpha)\left[2\sum_{n=1}^{\infty}\frac{J_1(\eta\beta_n)}{\beta_n J_0(\beta_n)} + \eta\right] + 8\sum_{n=1}^{\infty}\frac{\beta_n J_1(\eta\beta_n)}{J_0(\beta_n)} B(\xi, 4\beta_n^2, 0). \quad (35)$$

As it is proved in the Appendix, the following identity holds:

$$\sum_{n=1}^{\infty}\frac{J_1(\eta\beta_n)}{J_0(\beta_n)\beta_n} = \begin{cases} -\dfrac{\eta}{2} & 0 \leqslant \eta < 1 \\ 0 & \eta = 1 \end{cases}. \quad (36)$$

Therefore, equation (35) can be rewritten as

$$\chi(\eta,\xi) = \begin{cases} 8\displaystyle\sum_{n=1}^{\infty}\frac{\beta_n J_1(\eta\beta_n)}{J_0(\beta_n)} B(\xi, 4\beta_n^2, 0) & 0 \leqslant \eta < 1 \\ -F(4r_0^2\xi/\alpha) & \eta = 1 \end{cases}. \quad (37)$$

It is easily verified that equations (29) and (37) satisfy Fourier's law, i.e. $\chi = -\partial\vartheta/\partial\eta$, in the open set $0 \leqslant \eta < 1$.

If the local equilibrium hypothesis holds, any sufficiently small volume element of the solid can be considered in stable equilibrium, so that the entropy flux density is given by $\mathbf{q}/T$ [18]. Therefore, the local entropy balance equation can be written as

$$\nabla\cdot\left(\frac{\mathbf{q}}{T}\right) + \rho\frac{\partial s}{\partial t} = \sigma \quad (38)$$

where σ is the local entropy production rate per unit volume.

The local equilibrium hypothesis and the assumption that the mass density is constant imply that the relation $du = T\,ds$ holds locally. As a consequence, equations (2) and (33) yield

$$\sigma = -\frac{1}{T^2}\mathbf{q}\cdot\nabla T. \quad (39)$$

It is easily proved that the second law and the local equilibrium hypothesis imply that σ must be non-negative [13, 18].

On account of equations (8) and (39), within the cylinder, the entropy production rate per unit volume can be expressed as

$$\sigma = -\frac{q_0^2}{kT^2}\chi\frac{\partial\vartheta}{\partial\eta}. \quad (40)$$

Indeed, the local equilibrium hypothesis is conceivable if, and only if, at every instant of time and at every spatial position, χ and $\partial\vartheta/\partial\eta$ either vanish or have opposite sign. If ϑ and χ are such that at some instant of time either χ is positive and ϑ increases with η or χ is negative and ϑ decreases with η, the local equilibrium hypothesis is violated.

CONSTANT BOUNDARY HEAT FLUX

In this section, equations (27), (28) and (34) are employed to determine the distributions of the dimensionless temperature and of the dimensionless heat flux density, in the case of a constant boundary heat flux. At time $t = 0$, the boundary heat flux changes instantaneously from 0 to a non-vanishing constant value, so that $F(t) = U(t)$. Therefore, equation (11) yields $\Phi(\xi,\Lambda) = U(\xi) + \Lambda\delta(\xi)$ and equation (27) leads to the following expression of $B(\xi,\omega,\Lambda)$:

$$B(\xi,\omega,\Lambda) = \begin{cases} (\xi+\Lambda)U(\xi) & \omega = 0 \\ \left[\dfrac{1}{\omega} + \left(\Lambda - \dfrac{1}{\omega}\right)e^{-\omega\xi}\right]U(\xi) & \omega \neq 0 \end{cases}. \quad (41)$$

On account of equations (22) and (41), equation (28) can be rewritten as

$$\vartheta(\eta,\xi) = U(\xi)\left[8\xi + 2\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\beta_n^2} + 2\Lambda^2\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\beta_n^2\sqrt{1-16\Lambda\beta_n^2}}(\mu_n^2 e^{\lambda_n\xi} - \lambda_n^2 e^{\mu_n\xi})\right]. \quad (42)$$

As it is proved in the Appendix, the identity

$$\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\beta_n^2} = \frac{\eta^2}{4} - \frac{1}{8} \quad (43)$$

holds, so that equation (42) can be rewritten as

$$\vartheta(\eta,\xi) = U(\xi)\times\left[8\xi + \frac{2\eta^2-1}{4} + 2\Lambda^2\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\beta_n^2\sqrt{1-16\Lambda\beta_n^2}}\times(\mu_n^2 e^{\lambda_n\xi} - \lambda_n^2 e^{\mu_n\xi})\right]. \quad (44)$$

Equations (34) and (41) yield

$$\chi(\eta,\xi) = -U(\xi)\times\left[\eta + 2\Lambda\sum_{n=1}^{\infty}\frac{J_1(\eta\beta_n)}{J_0(\beta_n)\beta_n\sqrt{1-16\Lambda\beta_n^2}}\times(\mu_n e^{\lambda_n\xi} - \lambda_n e^{\mu_n\xi})\right]. \quad (45)$$

If Fourier's law holds, i.e. in the limit $\Lambda \to 0$, equations (29) and (37) can be employed. Therefore, on account of equations (41) and (43), one obtains

$$\vartheta(\eta,\xi) = U(\xi)\left[8\xi + \frac{2\eta^2-1}{4} - 2\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{\beta_n^2 J_0(\beta_n)}e^{-4\beta_n^2\xi}\right]. \quad (46)$$

$$\chi(\eta,\xi) = -U(\xi)\left[\eta + 2\sum_{n=1}^{\infty}\frac{J_1(\eta\beta_n)}{\beta_n J_0(\beta_n)}\,\mathrm{e}^{-4\beta_n^2\xi}\right]. \tag{47}$$

It is easily verified that equation (46) agrees with the solution of Fourier's equation obtained in Carslaw and Jaeger [19] for a solid cylinder with an uniform temperature distribution at $t = 0$ and a constant boundary heat flux for $t > 0$.

For values of ξ much greater than 1, exponentially decaying terms can be neglected in equation (44). Therefore, a large-time approximation of the dimensionless temperature distribution is given by

$$\vartheta(\eta,\xi) = 8\xi + \frac{2\eta^2 - 1}{4}. \tag{48}$$

Equation (48) shows that, for large times, the dimensionless temperature field predicted by the hyperbolic heat conduction equation tends to coincide with that predicted by Fourier's equation. More precisely, at a given position, the difference between the value of the dimensionless temperature evaluated by Fourier's equation and that evaluated by the hyperbolic equation of heat conduction tends to zero when $\xi \to +\infty$.

In Fig. 2, the evolution of ϑ at $\eta = 0$ for $\Lambda = 0.1$ is compared with the evolution of ϑ at the same point for $\Lambda = 0$. This plot shows that, if $\Lambda = 0.1$, the value of ϑ at $\eta = 0$ is zero for $\xi < \sqrt{0.1}/2 \approx 0.158$. Indeed, the time necessary for the thermal wave to travel from the boundary of the cylinder to the axis is $t = L\sqrt{\tau/\alpha}$, that is $\xi = \sqrt{\Lambda}/2$. Moreover, this figure shows that the plot ϑ vs ξ for $\Lambda = 0.1$ presents two singularities. The first is when $\xi = \sqrt{0.1}/2$, i.e. when the wavefront of the temperature field reaches the axis of the cylinder for the first time. The second singularity occurs when $\xi = 3\sqrt{0.1}/2 \approx 0.474$, i.e. when the wavefront of the temperature field, after having been reflected from the boundary of the cylinder, reaches the axis for the second time.

The singularities of the temperature field are due to the existence of a step change of the boundary heat flux. Indeed, as it is shown in Fig. 3, the instantaneous finite change of the boundary heat flux produces a sharp wavefront both for ϑ and for χ. As it is shown in Fig. 4, when the sharp wavefront reaches the axis, a singularity arises. If the finite change of boundary heat flux were performed with continuity, the wavefront would not be sharp and no singularity would appear.

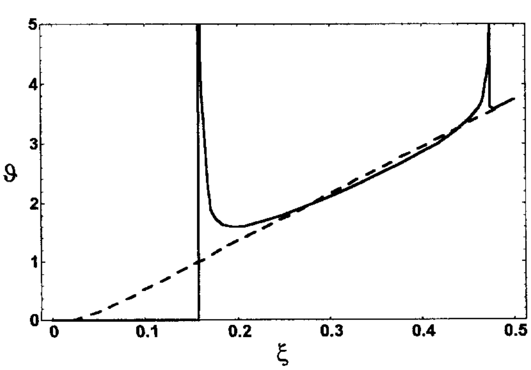

Fig. 2. Plots of ϑ vs ξ for $\eta = 0$ in the case of a constant boundary heat flux; the solid line refers to $\Lambda = 0.1$ and the dashed line to $\Lambda = 0$.

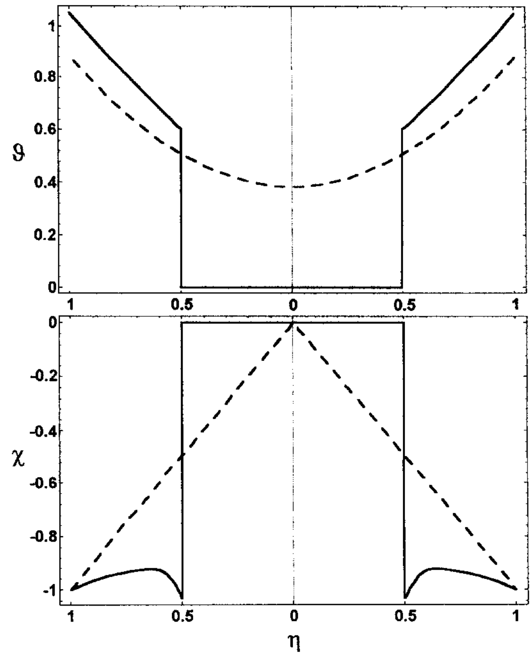

Fig. 3. Plots of ϑ and χ vs η for $\xi = \sqrt{0.1}/4$ in the case of a constant boundary heat flux; the solid line refers to $\Lambda = 0.1$ and the dashed line to $\Lambda = 0$.

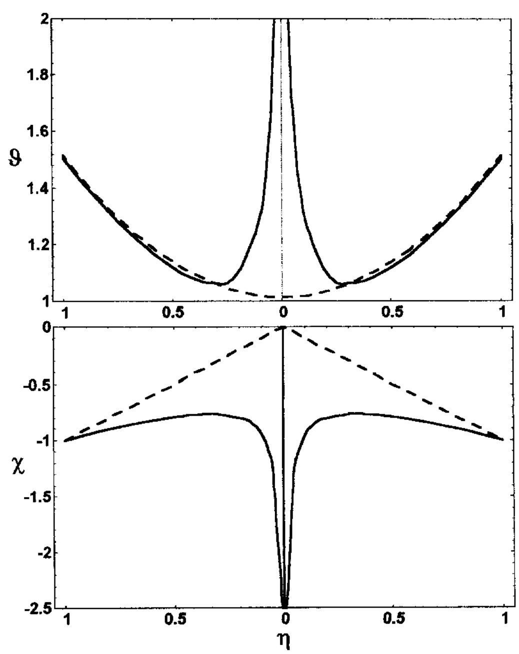

Fig. 4. Plots of ϑ and χ vs η for $\xi = \sqrt{0.1}/2$ in the case of a constant boundary heat flux; the solid line refers to $\Lambda = 0.1$ and the dashed line to $\Lambda = 0$.

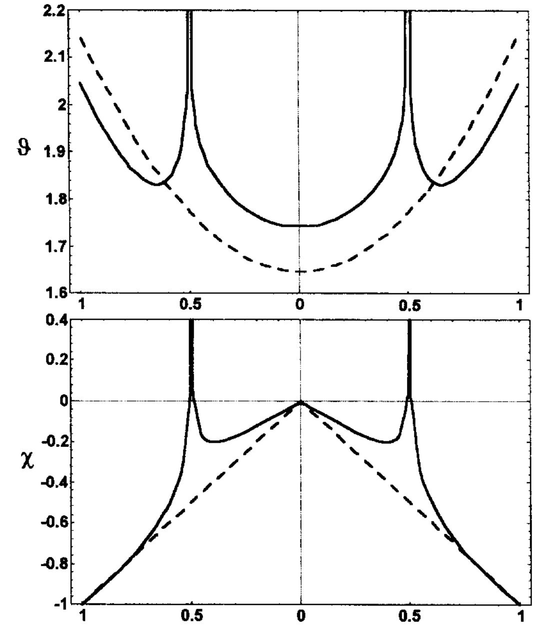

Fig. 5. Plots of ϑ and χ vs η for $\xi = 3\sqrt{0.1}/4$ in the case of a constant boundary heat flux; the solid line refers to $\Lambda = 0.1$ and the dashed line to $\Lambda = 0$.

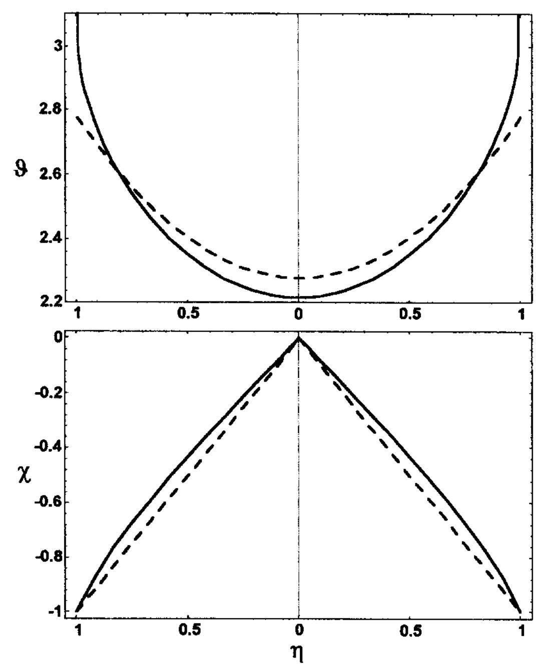

Fig. 6. Plots of ϑ and χ vs η for $\xi = \sqrt{0.1}$ in the case of a constant boundary heat flux; the solid line refers to $\Lambda = 0.1$ and the dashed line to $\Lambda = 0$.

Figure 5 shows that, for $\xi = 3\sqrt{0.1}/4$ and $\Lambda = 0.1$, both ϑ and χ are singular on the surface $\eta = 0.5$. Indeed, for $\sqrt{\Lambda}/2 < \xi < \sqrt{\Lambda}$ a singular surface $\eta = 2\xi/\sqrt{\Lambda} - 1$ travels from the axis of the cylinder to the boundary, this surface is singular both for the dimensionless temperature and for the dimensionless heat flux. As it is shown in Fig. 6, for $\xi = \sqrt{0.1}$ and $\Lambda = 0.1$ the dimensionless temperature ϑ is singular on the boundary. Figures 3–6 reveal that, if $\Lambda = 0$, the heat flux distribution can be considered as stationary for $\xi > \sqrt{0.1}/4$.

By employing the method described in the previous section, the compatibility of the distributions of ϑ and χ reported in Figs. 3–6 with the hypothesis of local thermodynamic equilibrium can be analysed. In particular, Figs. 3 and 6 show that $\partial\vartheta/\partial\eta$ and χ have an opposite sign in agreement with the local equilibrium hypothesis. On the other hand, Figs. 4 and 5 show that in the case $\Lambda = 0.1$, both for $\xi = \sqrt{0.1}/2$ in the region $\eta \leqslant 0.25$ and for $\xi = 3\sqrt{0.1}/4$ in the region $0.5 \leqslant \eta \leqslant 0.65$, $\partial\vartheta/\partial\eta$ and χ have the same sign. This circumstance is in contrast with the local equilibrium hypothesis.

EXPONENTIALLY DECAYING HEAT FLUX PULSE

In this section, equations (27), (28) and (34) are employed to study the thermal wave propagation of an exponentially decaying heat flux pulse prescribed at the boundary of the cylinder. Let us consider a time-varying boundary heat flux such that $F(t)$ is given by

$$F(t) = \frac{t^2}{16t_p^2}\,e^{-t/t_p}. \tag{49}$$

A plot of $F(t)$ is shown in Fig. 7. If one defines the dimensionless parameter

$$\Pi = \frac{\alpha t_p}{4r_0^2} \tag{50}$$

equation (49) can be rewritten as

$$F(4r_0^2\xi/\alpha) = \frac{\xi^2}{16\Pi^2}\,e^{-\xi/\Pi}. \tag{51}$$

Moreover, equations (11) and (51) yield

$$\Phi(\xi,\Lambda) = \frac{e^{-\xi/\Pi}}{16\Pi^3}\,[(\Pi-\Lambda)\xi^2 + 2\Lambda\Pi\xi]. \tag{52}$$

Equation (51) implies that the heat supplied to the

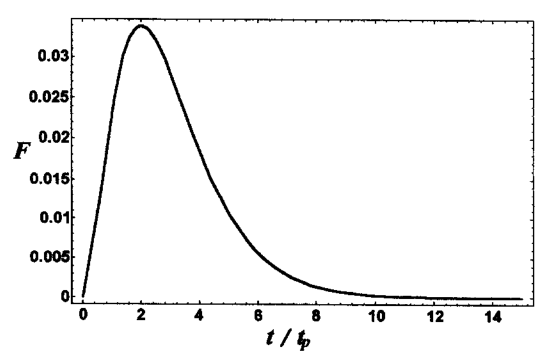

Fig. 7. Plot of F vs t/t_p in the case of an exponentially decaying boundary heat flux.

cylinder per unit length in the time interval $[0, +\infty]$ is finite and can be expressed as

$$Q_\infty = 2\pi r_0 q_0 \int_0^{+\infty} F(t')\,dt' = \frac{\pi r_0^3 q_0}{\alpha}\Pi. \quad (53)$$

Since $F(t)$ tends to zero in the limit $t \to +\infty$, the temperature field is expected to reach a uniform distribution in this limit. On account of equation (53), a simple energy balance allows the evaluation of the asymptotic value of the dimensionless temperature, namely

$$\lim_{t\to+\infty} \vartheta = \lim_{t\to+\infty} k\frac{T-T_0}{r_0 q_0} = \frac{k}{r_0 q_0}\frac{Q_\infty}{\pi r_0^2 \rho c} = \Pi. \quad (54)$$

As a consequence of equations (27) and (52), $B(\xi, \omega, \Lambda)$ is given by

$$B(\xi,\omega,\Lambda) = \frac{\Pi(\omega\Lambda-1)}{8(\omega\Pi-1)^3}e^{-\omega\xi} + \frac{e^{-\xi/\Pi}}{16\Pi^2(\omega\Pi-1)^3}[(\omega\Pi-1)^2(\Pi-\Lambda)\xi^2 + 2\Pi^2(\omega\Pi-1)(\omega\Lambda-1)\xi - 2\Pi^3(\omega\Lambda-1)]. \quad (55)$$

Moreover, equation (51) implies that

$$\int_0^{\xi} F(4r_0^2\xi'/\alpha)\,d\xi' = \frac{\Pi}{8} - \frac{e^{-\xi/\Pi}}{16\Pi}(\xi^2+2\Pi\xi+2\Pi^2). \quad (56)$$

Equations (28), (34), (51), (55) and (56) can be employed to evaluate the dimensionless temperature field and the dimensionless heat flux density distribution inside the cylinder, namely

$$\vartheta(\eta,\xi) = \Pi - \frac{e^{-\xi/\Pi}}{2\Pi}(\xi^2+2\Pi\xi+2\Pi^2) - 8\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)\sqrt{1-16\Lambda\beta_n^2}} \times [B(\xi,-\lambda_n,\Lambda) - B(\xi,-\mu_n,\Lambda)] \quad (57)$$

$$\chi(\eta,\xi) = -\eta\frac{\xi^2}{16\Pi^2}e^{-\xi/\Pi} + 2\sum_{n=1}^{\infty}\frac{J_1(\eta\beta_n)}{\beta_n J_0(\beta_n)\sqrt{1-16\Lambda\beta_n^2}} \times [\lambda_n B(\xi,-\lambda_n,\Lambda) - \mu_n B(\xi,-\mu_n,\Lambda)]. \quad (58)$$

Equation (22) implies that, for every n, $\mathrm{Re}\{\lambda_n\} < 0$ and $\mathrm{Re}\{\mu_n\} < 0$, so that equation (55) ensures that, for $\xi \to +\infty$, $B(\xi, -\lambda_n, \Lambda) \to 0$ and $B(\xi, -\mu_n, \Lambda) \to 0$. Therefore, equation (57) is in agreement with equation (54).

If Fourier's law holds, i.e. in the limit $\Lambda \to 0$, equations (29) and (37) hold. Therefore, by employing equations (51) and (56), one obtains

$$\vartheta(\eta,\xi) = \Pi - \frac{e^{-\xi/\Pi}}{2\Pi}(\xi^2+2\Pi\xi+2\Pi^2) + 8\sum_{n=1}^{\infty}\frac{J_0(\eta\beta_n)}{J_0(\beta_n)}B(\xi,4\beta_n^2,0) \quad (59)$$

$$\chi(\eta,\xi) = \begin{cases} 8\sum_{n=1}^{\infty}\frac{\beta_n J_1(\eta\beta_n)}{J_0(\beta_n)}B(\xi,4\beta_n^2,0) & 0 \leqslant \eta < 1 \\ -\frac{\xi^2}{16\Pi^2}e^{-\xi/\Pi} & \eta = 1 \end{cases}. \quad (60)$$

In Figs. 8–11, the distributions of ϑ and χ as functions of η are reported for $\Lambda = 1$, $\Lambda = 0$ and $\Pi = 0.05$, with the values of the dimensionless time, $\xi = 1/4$, $\xi = 1/2$, $\xi = 3/4$ and $\xi = 1$. As it is shown in the following, in the case $\Lambda = 1$, all these figures exhibit violations of the local equilibrium hypothesis. Figure 8 refers to $\xi = 1/4$ and shows that, if $\Lambda = 1$, the thermal wave has not yet reached the axis. Moreover, this figure reveals that, for $\xi = 1/4$ and $\Lambda = 1$, χ and $\partial\vartheta/\partial\eta$ have the same sign in the range $0.7 \leqslant \eta \leqslant 1$. Figure 9 refers to $\xi = 1/2$ and shows that, if $\Lambda = 1$, the thermal wave has reached the axis, the dimensionless heat flux vector is directed inward for $\eta \leqslant 0.65$, while its direction is outward for $0.65 \leqslant \eta < 1$. This figure shows that, for $\xi = 1/2$ and $\Lambda = 1$, χ and $\partial\vartheta/\partial\eta$ have the same sign in the range $0.15 \leqslant \eta \leqslant 0.65$. Figure 10 shows that, for $\xi = 3/4$ and $\Lambda = 1$, the thermal wave is travelling from the axis to the boundary of the cylinder. This figure

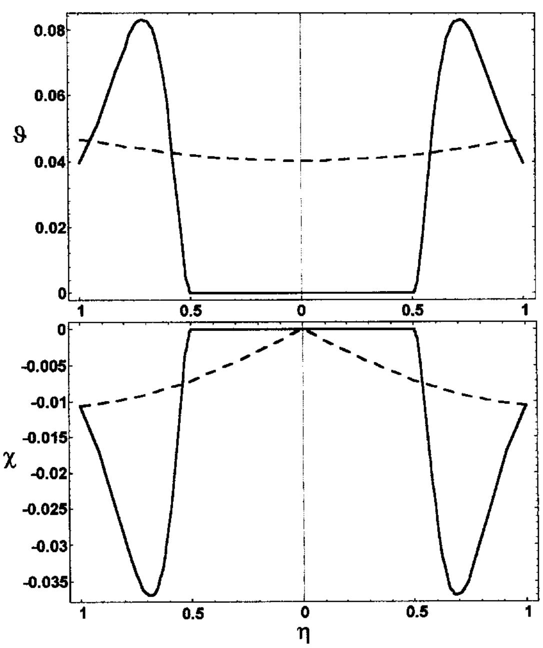

Fig. 8. Plots of ϑ and χ vs η for $\xi = 1/4$ in the case of an exponentially decaying boundary heat flux with $\Pi = 0.05$; the solid line refers to $\Lambda = 1$ and the dashed line to $\Lambda = 0$.

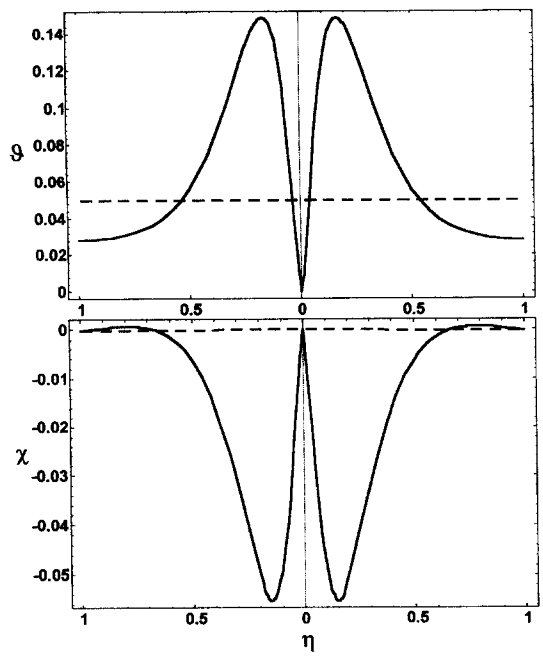

Fig. 9. Plots of ϑ and χ vs η for $\xi = 1/2$ in the case of an exponentially decaying boundary heat flux with $\Pi = 0.05$; the solid line refers to $\Lambda = 1$ and the dashed line to $\Lambda = 0$.

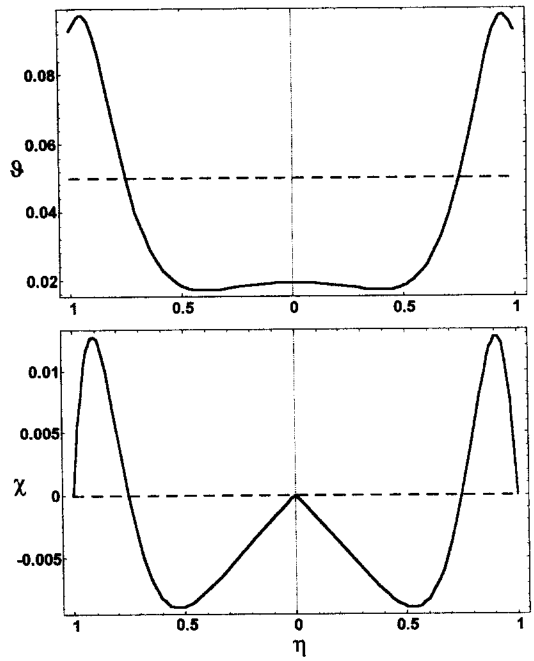

Fig. 11. Plots of ϑ and χ vs η for $\xi = 1$ in the case of an exponentially decaying boundary heat flux with $\Pi = 0.05$; the solid line refers to $\Lambda = 1$ and the dashed line to $\Lambda = 0$.

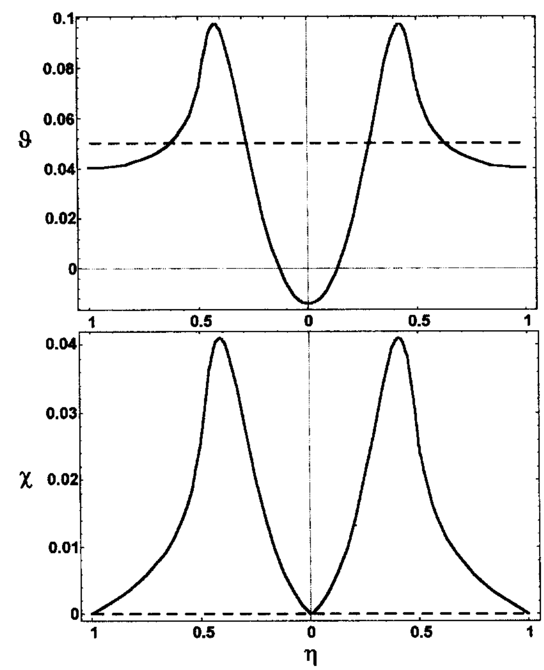

Fig. 10. Plots of ϑ and χ vs η for $\xi = 3/4$ in the case of an exponentially decaying boundary heat flux with $\Pi = 0.05$; the solid line refers to $\Lambda = 1$ and the dashed line to $\Lambda = 0$.

reveals that χ and $\partial\vartheta/\partial\eta$ have the same sign in the range $\eta \lesssim 0.4$. Moreover, if $\xi = 3/4$ and $\Lambda = 1$, the dimensionless temperature ϑ is negative for $\eta \lesssim 0.15$. As a consequence, for $\eta \lesssim 0.15$, if $q_0 > 0$ (i.e. if the cylinder is heated) the temperature is lower than its initial value T_0, while if $q_0 < 0$ (i.e. if the cylinder is cooled) the temperature is higher than its initial value T_0. This effect does not represent a violation of the second law, since the local equilibrium scheme cannot be applied in this case. Finally, Fig. 11 shows that, for $\xi = 1$ and $\Lambda = 1$, the dimensionless heat flux vector is directed inward for $\eta \lesssim 0.75$, while its direction is outward for $0.75 \lesssim \eta < 1$. This figure shows that, for $\xi = 1$ and $\Lambda = 1$, χ and $\partial\vartheta/\partial\eta$ have the same sign both in the range $\eta \lesssim 0.4$ and in the range $0.75 \lesssim \eta \lesssim 0.95$. Figures 8–11 show that, for $\Lambda = 0$, the large-time uniform distributions $\vartheta = 0.05$ and $\xi = 0$ are attained for $\xi \gtrsim 1/2$.

CONCLUSIONS

The non-stationary heat conduction within a solid cylinder with a uniform and time-varying boundary heat flux has been studied for a solid material with a non-vanishing thermal relaxation time. The hyperbolic heat conduction equation together with its initial and boundary conditions has been written in a dimensionless form. The dimensionless temperature field has been determined by the Laplace transform method. Then, the dimensionless heat flux has been obtained by employed the local energy balance equation. The case of validity of Fourier's law is obtained in the limit of a vanishing thermal relaxation time. The general solution is employed to analyse the case of a constant boundary heat flux and the case of an exponentially decaying boundary heat flux. For a constant boundary heat flux, both the temperature field and the heat flux density distribution are affected by singularities. These singularities are due to the instantaneous change of boundary heat flux which is prescribed for

$t = 0$. It has been shown that, both in the case of a constant boundary heat flux and in the case of an exponentially decaying boundary heat flux, violations of the local equilibrium hypothesis can occur. The violations of the local equilibrium hypothesis have been found by checking the sign of the entropy production rate per unit volume. Indeed, if at some instant of time in some region of the solid, the sign of the entropy production rate is negative, then the local equilibrium hypothesis cannot be applied.

Acknowledgements—The authors are grateful to Professor E. Zanchini for helpful discussions on some topics treated in this paper. Financial support has been provided by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (MURST).

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APPENDIX

A function $g(\eta)$ can be expanded by a series of Bessel functions. In particular, the following relation holds [20]

$$g(\eta) = a_0 + \sum_{n=1}^{\infty} a_n J_0(\beta_n \eta) \tag{A1}$$

where β_n is the nth root of $J_1(y) = 0$, while a_0 and a_n are given by

$$a_0 = 2\int_0^1 \eta g(\eta)\,\mathrm{d}\eta \quad a_n = \frac{2}{[J_0(\beta_n)]^2}\int_0^1 \eta g(\eta) J_0(\beta_n \eta)\,\mathrm{d}\eta. \tag{A2}$$

Let us consider the expansion of $g(\eta) = \eta^2$. Equation (A2) can be rewritten as

$$a_0 = \frac{1}{2} \quad a_n = \frac{2}{\beta_n^4 [J_0(\beta_n)]^2}\int_0^{\beta_n} y^3 J_0(y)\,\mathrm{d}y. \tag{A3}$$

On account of the identity [20]

$$\int y^3 J_0(y)\,\mathrm{d}y = (y^3 - 4y)J_1(y) + 2y^2 J_0(y) \tag{A4}$$

equations (A1) and (A3) yield

$$\eta^2 - \frac{1}{2} = 4\sum_{n=1}^{\infty} \frac{J_0(\beta_n \eta)}{\beta_n^2 J_0(\beta_n)}. \tag{A5}$$

Since the first derivative of function J_0 is given by $-J_1$, equation (A5) leads to the identity

$$\eta = -2\sum_{n=1}^{\infty} \frac{J_1(\beta_n \eta)}{\beta_n J_0(\beta_n)}. \tag{A6}$$